Notes on Groups and Geometry, 1978-1986
by Steven H. Cullinane

Typewritten notes collected in a 40-page PDF document.

CONTENTS

01 1978-??-?? "An invariance of symmetry" Research announcement.
03 1978-12-?? "Orthogonality of Latin squares viewed as as skewness of lines"
04 1981-11-05 "Patterns invariant, modulo rigid motions, under groups of discontinuous transformations: two examples"
05 1981-12-24 "Solid symmetry"- "Motions of each cell induce motions of the entire pattern."
06 1982-05-12 "Map systems" Definition.
07 1982-06-12 "Inscapes" Definition.
08 1982-06-12 "A symplectic array"
09 1982-09-22 "Inscapes II" Generalized definition.
10 1982-12-27 "Group scores" Definition.
11 1983-05-31 "Decompositions of group enveloping algebras"
12 1983-06-21 "An invariance of symmetry" Group actions on a 4x4x4 cube.
13 1983-08-04 "Group identity algebras" Definition.
14 1982-08-26 "Transformations over a bridge" Definition and problem.
17 1983-11-08 "Compound groups" Definition and problem.
18 1983-11-10 "Group compounds" Definition and problem.
19 1983-11-27 "Table groups" Definition and problem.
20 1984-01-05 "Linear operators in geometric function spaces" Definition, theorem, problem.
21 1984-08-15 "Diamonds and whirls" Blocks illustrate group actions.
22 1984-08-25 "Affine groups on small binary spaces" Theorem.
23 1985-03-26 "Visualizing GL(2,p)" Example on a 3x3 array.
24 1985-04-05 "GL(2,3) actions on a cube"
25 1985-04-05 "Group actions on partitions" Definition and problem.
26 1985-04-28 "Generating the octad generator"
27 1985-08-22 "Symmetry invariance under M12" Theorem.
28 1985-11-17 "A group bridge" Definition and problem.
29 1986-12-11 "Dynamic and algebraic compatibility of groups" Problems.
30 1986-01-11 "Geometry of partitions II" A foray into analysis.
31 1986-02-04 "Inscapes III: PG(2,4) from PG(3,2)"
32 1986-02-20 "The relativity problem in finite geometry"
33 1986-03-31 "Group topologies" Definition and problems.
34 1986-04-26 "Picturing the smallest projective 3-space"
35 1986-05-03 "A linear complex related to M24"
36 1986-05-28 "The 2-subsets of a 6-set are the points of a PG(3,2)"
38 1986-06-11 "An outer automorphism of S6 related to M24"
39 1986-07-03 "Picturing outer automorphisms of S6"
40 1986-07-11 "Inscapes IV: Inner and outer group actions"

Web page URL:  http://finitegeometry.org/sc/gen/typednotes.html
AN INVARIANCE OF SYMMETRY

BY STEVEN H. GULLINANE

We present a simple, surprising, and beautiful combinatorial invariance of geometric symmetry, in an algebraic setting.

DEFINITION. A delta transform of a square array over a 4-set is any pattern obtained from the array by a 1-to-1 substitution of the four diagonally-divided two-color unit squares for the 4-set elements.

EXAMPLES. 

THEOREM. Every delta transform of the Klein group table has ordinary or color-interchange symmetry, and remains symmetric under the group G of 322,560 transformations generated by combining permutations of rows and columns with permutations of quadrants.

EXAMPLE. 

PROOF (Sketch). The Klein group is the additive group of GF(4); this suggests we regard the group's table T as a matrix over that field. So regarded, T is a linear combination of three (0,1)-matrices that indicate the locations, in T, of the 2-subsets of field elements. The structural symmetry of these matrices accounts for the symmetry of the delta transforms of T, and is invariant under G.

All delta transforms of the 45 matrices in the algebra generated by the images of T under G are symmetric; there are many such algebras.

THEOREM. If 1 \leq m \leq n^2 + 2, there is an algebra of 4

2n x 2n matrices over GF(4) with all delta transforms symmetric.

An induction proof constructs sets of basis matrices that yield the desired symmetry and ensure closure under multiplication.

REFERENCE

S. H. Gullinane, Diamond theory (preprint).
Steven Hamilton Collinane

Symmetry invariance in a diamond ring. Preliminary report.

We regard the four-diamond figure $D$ as a $4 \times 4$ array of 2-color tiles such as $\mathcal{C}$. Let $G$ be the group of $322,560$ permutations of these $16$ tiles generated by arbitrarily mixing random permutations of rows and of columns with random permutations of the four $2 \times 2$ quadrants.

**Theorem:** Every $G$-image of $D$ has some ordinary or color-interchange symmetry.

**Example:**

$D$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$ $\mathcal{C}$

$D_g = (\text{where } g \in G \text{ is a product of two disjoint } 2\text{-cycles})$.

Note that $D_g$ has rotational color-interchange symmetry like that of the famous yin-yang symbol.

**Remarks:** $G$ is isomorphic to the affine group on $\mathbb{F}_2$ ($\text{GF}(2)$). The $35$ structures of the $840 \times 35 \times 24$ $G$-images of $D$ are isomorphic to the $35$ lines in the $3$-dimensional projective space over $\text{GF}(2)$; orthogonality of structures corresponds to skewness of lines. We can define sums and products so that the $G$-images of $D$ generate an ideal $(1,024$ patterns characterized by all horizontal or vertical "cuts" being uninterrupted) of a ring of $4,096$ symmetric patterns. There is an infinite family of such "diamond" rings, isomorphic to rings of matrices over $\text{GF}(4)$. (Received October 31, 1978.)
STEVEN H. CULLINANE

Orthogonality of Latin squares viewed as skewness of lines.

Shown below is a way to embed the six order-4 Latin squares that have orthogonal Latin mates in a set of 35 arrays so that orthogonality in the set of arrays corresponds to skewness in the set of 35 lines of \(PG(3, 2)\).

Each array yields a 3-set of diagrams that show the lines separating complementary 2-subsets of \{0, 1, 2, 3\}; each diagram is the symmetric difference of the other two. The 3-sets of diagrams correspond to the lines of \(PG(3, 2)\). Two arrays are orthogonal iff their 3-sets of diagrams are disjoint, i.e., iff the corresponding lines of \(PG(3, 2)\) are skew.

This is a new way of viewing orthogonality of Latin squares, quite different from their relationship to projective planes.

**PROBLEM:** To what extent can this result be generalized? (Dec. 1978)

In fig. 1, rigid motions of each cell in a pattern induce rigid motions of the entire pattern. In fig. 3, permutations of cells produce various sectional views of the same (modulo rigid motions) infinite plane pattern. These permutations are derived, as in fig. 2, from motions of a cube.
Steven H. Cullinane  

In pattern A, motions of each cell induce motions of the entire pattern; likewise in B.

Definition: Suppose every map $\Phi$ into a given ring $M$ can be written as $\Phi = \sum_{i=1}^{n} e_i(\alpha_i \circ \phi)$, where $N$ is a fixed positive integer, the $e_i$ are fixed elements of $M$, and the $\alpha_i$ are fixed functions from $M$ to $P$, a proper subset of $M$.

Let $A = \{\alpha_1, \ldots, \alpha_A\}$, $O = \{e_1, \ldots, e_N\}$.

The quintuple $(M, A, P, O, N)$ is a map system.

Example: Using hexadecimal labels for the elements of $GF(16)$, let $(M, A, P, O, N) = (GF(16), A, \{0,7,E,9\}, \{6,8,F\}, 3)$, where the functions in $A$ are specified by giving inverse images:

\[
\begin{pmatrix}
\{0,1,3,2\} & \{0\} & \{0,4,C,8\} & \{0\} & \{0,5,F,A\} & \{0\} \\
\{4,5,7,6\} & \{7\} & \{1,5,D,9\} & \{9\} & \{1,4,E,B\} & \{E\} \\
\{C,D,F:E\} & \{E\} & \{3,7,F,B\} & \{7\} & \{3,6,C,9\} & \{9\} \\
\{8,9,B,A\} & \{9\} & \{2,6,E,A\} & \{E\} & \{2,7,D,8\} & \{7\}
\end{pmatrix} = (\alpha_1, \alpha_2, \alpha_3).
\]

(Multiplication in $GF(16)$ is here defined via the irreducible polynomial $x^4 + x + 1$.)

Remarks: Harmonic analysis allows a complicated map to be broken down into, or built up from, simpler maps.

Map systems are a different means to the same end.

Query: What is known about such systems?
Definition: Let $R$ be an $n$-ary symmetric relation on a set of $t$ subsets of a $t$-set, where $n < t = uv$, for positive integers $n, t, u, v$. Represent each of the $t$ subsets by the 1's in a $u \times v$ array $a_i$ over $GF(2)$, where $1 \leq i \leq t$. An *inscape* of $R$ is a $u \times v$ array $A$ of the $a_i$ such that $R$ is true for $n$ of the $a_i$ (that is, for the subsets represented by these $a_i$) if and only if the arrangement of the $a_i$ within $A$ is the same as the arrangement of the 1's in some $a_i \neq \phi$.

Examples: (Light and dark represent 0's and 1's.)

Remarks: *Inscape* are useful for visualizing relations in certain finite geometries. The above examples, for instance, illustrate relations among the 15 hyperplanes of $PG(3,2)$ and among the 15 lines fixed under a particular symplectic polarity of $PG(3,2)$.

Query: What is known about combinatorial systems of this sort?


The 11x11 array below is formed by adding (light = 0, dark = 1, 1+1 = 0) the 10 nonempty squares in the first column to the 10 nonempty squares in the first row. These squares represent the 10 pairs of lines interchanged under a particular symplectic polarity of PG(3,2). The array is of interest for several reasons:

1) It serves to illustrate an elementary, but useful, way of constructing a complicated combinatorial object from simpler objects: make an addition table. (Closure is not essential.)

2) Properties of arrays thus formed as addition tables may be of some use in the study of 10x10 Latin squares.

3) Since each of the 121 4x4 squares below represents a set of points in a finite projective space, the array may serve to illustrate or to suggest properties of such spaces.
Steven H. Cullinane  

Given a set $X$ of points, certain families of subsets of $X$ may have, as families, some property $s$. (Example: the families of spheres that are concentric.) It may be that we can associate to each point of $X$ a subset of $X$, via an injection $f: X \rightarrow 2^X$, in such a way that the $f$-image, in turn, of this subset of $X$ (i.e., the family of $f$-images of its points) is in fact one of the families of subsets of $X$ that have property $s$.

If the map $f$ gives rise in this way to the set $S$ of all such $s$-families, we can write, in a cryptic but concise way, $S = f(f(X))$, and say that $f$ is an incape of $S$.

Query: What known results can be stated, after the appropriate definition of $S$, in the form “There exists an incape of $S$”?

Addendum of Oct. 10, 1982. A more precise definition:

Let $X$ be a non-empty set. Let $P(X)$ denote the set of all subsets of $X$. Let $S \subseteq P(P(X))$. Suppose there exists an injection $f: X \rightarrow P(X)$ such that, for any $\sigma \in P(P(X))$, $\sigma \in S$ if and only if $\exists x \in X$ such that $\sigma = f(f(x)) = \{f(y) \mid y \in f(x)\}$.

Then $f$ is an incape of $S$.

This notion arises naturally in studying the action of a symplectic polarity in a projective space. One of course wonders whether it has arisen previously in any other context.
Definition: Let $G_1$ be a finite permutation group. Represent $G_1$ as a group of permutation matrices over $GF(2)$, the two-element Galois field, and let $V_1 = V_1(G_1)$ denote the enveloping algebra\(^a\) of $G_1$. Suppose there exist subalgebras $V_2$ and $V_3$ of $V_1$ such that

a) $V_3$ is a transversal of the additive cosets of $V_2$ in $V_1$, and

b) $V_3$ is the enveloping algebra of a subgroup $G_2$ of $G_1$.

Then $(G_1, V_1, V_2, V_3, G_2)$ is a group score.

Example (Light and dark represent 0's and 1's):

![Matrix representation]

This array $A$ of matrices is a group score in which

$G_1 = S_3$,

$V_1 = A$,

$V_2$ = the first row of $A$,

$V_3$ = the first column of $A$, and

$G_2$ = the permutation matrices in the first column.

Note that in the example neither $V_2$ nor $V_3$ is an ideal of $V_1$.

Problem: What group scores exist?

\(^a\) See Flath, AMS abstract 797-17-88, the related 797-20-130, and H. Weyl, The classical groups, 2nd ed. (1946), Princeton, p. 79

Notation:

Let $G$ be an abstract group, $H$ a subgroup of $G$. Let $\rho: G \to M$ be a representation of $G$ as a group $M$ of invertible endomorphisms of an $R$-module $V$, where $R$ is a commutative ring with unity, and let $N \leq \rho(G)$. Denote the enveloping algebra of $M$ (i.e., the $R$-linear closure of $M$) by $E(M)$, or, more explicitly, by $(E(M),+,\cdot)$. For $a,b$ in $E(M)$ let $[a,b] = a \cdot b - b \cdot a$, and denote the resulting Lie algebra by $(E(N),+,\cdot)$. 

Query:

1. How can we relate decompositions of $(E(M),+,\cdot)$ to the structure of $G$?
   
   (In particular, when can we write $E(M)$ as a direct sum 
   \[ E(M) = E(N) \oplus A, \]
   
   where $A$ is a subalgebra of $E(N)$?)

2. How can we relate decompositions of $(E(X),+,\cdot)$, where $X = M$ or $N$, to the structure of $G$, when $M$ is nonabelian?
   
   (In particular, how are the Levi direct sum decompositions\(^\ast\)
   
   \[ E(M) = R(N) \oplus L(M) \]
   \[ E(N) = R(N) \oplus L(N), \]
   
   where $R(X)$ is the radical of $(E(X),+,\cdot)$ and $L(X) = E(X)/R(X)$ is a semisimple Lie subalgebra of $(E(X),+,\cdot)$, related to the structure of $G$?)

3. How should we restrict the natures of $G$, $H$, $\rho$, $M$, $R$, and $V$ in order to answer (1) or (2) above?

\(^\ast\) See A. I. Mal'cev, On semisimple subgroups of Lie groups (1944), AMS Translations, series 1, volume 9 (1962).
Theorem: There exists a triply transitive group $G$ of
1, 290, 157, 124, 640 permutations of the 64 subsquares of $B$ such that
every $G$-image of $B$ has a rigid-motion symmetry.

(The marking on each subsquare of $B$ is identical; each is symmetric
under reflection in its center.)

Proof: (sketch): We label the 64 cells of $B$ with the points of the
affine 6-space $A$ over $GF(2)$ in such a way that each hyperplane of $A$
is left invariant or is carried to its complement under a group $G$
of 6 rigid motions generated by reflections in midplanes of $B$.
We then define the group $G$ as the group of affine transformations of $A$.
Under $G$, as under $G$, the set of hyperplanes and hyperplane-complements
is left invariant. This symmetry of hyperplanes is then fairly easily
shown to underlie the remarkable invariance of symmetry of $B$.

(For a geometrically natural way to generate $G$ see AMS abstract 79T-A37.)
Steven H. Gullinana

Definition: Let \((S, *)\) and \((S, \circ)\) be groups with the same set \(S\) of element-symbols but with different group tables.

If there is at least one algebraic identity I expressing a nontrivial relationship between \(*\) and \(\circ\) then \((S, *, \circ)\) is a sort of algebra, which for lack of any other name we call a group identity algebra.

Example: Let \(S = \{a, b, c, e\}\) and let \(*\) and \(\circ\) be the operations + and \(\cdot\) (or juxtaposition) in the tables below.

\[
\begin{array}{c|cccc}
  + & a & b & c & e \\
  \\hline
  a & a & b & c & e \\
  b & a & b & c & e \\
  c & a & b & c & e \\
  e & a & b & c & e \\
\end{array}
\quad
\begin{array}{c|cccc}
  \cdot & a & b & c & e \\
  \\hline
  a & a & b & c & e \\
  b & a & b & c & e \\
  c & a & b & c & e \\
  e & a & b & c & e \\
\end{array}
\]

The following identity I holds. \(\forall x, y, z, w \in \{a, b, c, e\}\),

\[
((xy) + (zw)) + ((x + y)(z + w)) =
((xz) + (yw)) + ((x + z)(y + w)) =
((xw) + (yz)) + ((x + w)(y + z)).
\]

The dual identity \(I'\) obtained by interchanging + and \(\cdot\) also holds.

(Note that in this case I and \(I'\) state that certain algebraic forms are invariant under the action of the symmetric group on their indeterminates.)

Problem: Are there infinitely many finite group identity algebras?

(Note the word "nontrivial" in the definition.)
Steven H. Cullinane  

Let $(G, +, *)$ be the algebra (in the sense of universal algebra) with underlying set $G = \{e, a, b, c\}$ and operations as follows.

\[
\begin{array}{c|cccc}
+ & e & a & b & c \\
\hline
e & e & a & b & c \\
a & a & e & b & c \\
b & b & c & e & a \\
c & c & b & a & e \\
\end{array}
\quad
\begin{array}{c|cccc}
* & e & a & b & c \\
\hline
e & e & a & b & c \\
a & a & e & b & c \\
b & b & c & e & a \\
c & c & b & a & e \\
\end{array}
\quad
(We follow the notational conventions of writing $-$ as juxtaposition and letting $+$ preceed $-$ to avoid a proliferation of parentheses.)

(Such an algebra $(G, \triangledown, \star)$, where $(G, \triangledown)$ and $(G, \star)$ are groups, we call a bridge.)

Problem.

Part 1. What is the nature of the group $T$ generated by permutations of $G \times G$ of the form $t(p, q, r, s): (x, y) \rightarrow (px + qy, rx + sy) + (x, y)$ where $p, q, r, s \in G$?

Note 1.1. It appears that $T$ is isomorphic to a subgroup of the group of regular affine transformations of the affine $l$-space over $GF(2)$.

Note 1.2. We can have $t(p, q, r, s) = t(t(u, v, w))$ where $(p, q, r, s) \not\simeq (t(u, v, w))$ since, for instance, $\forall (x, y) \in G \times G$, $ax + ay = cx + cy$.

Part 2. Is there some reasonably simple algebraic expression over $(G, +, \cdot)$ for $t(p, q, r, s) \circ t(t(u, v, w))$?

Note 2.1. Not every member of $T$ can be written in the form $t(p, q, r, s)$. Example: $(t(a, c, c, e))^2$. 

14
Steven H. Cullinan
Steven E. Cullinane  

The 24 two-colored 2x2x2 cubes below represent the elements of the octahedral group O, which is viewed as acting in the same way on each of the 8 subcubes of any given 2x2x2 cube. The arrangement of the 24 colored cubes may be of some interest for its combinatorial properties.
Definition: Suppose a finite group \( G \) of order \( n \) can be represented as a group of permutations \( p_1, p_2, \ldots, p_n \) on \( m \) objects, where \( m < n \). Suppose further that we can take these \( m \) objects to be distinct elements \( g_1, g_2, \ldots, g_m \) of \( G \) in such a way that the \( n \) products

\[
(p_1(g_1))(p_1(g_2)) \cdots (p_1(g_m))
\]
\[
(p_2(g_1))(p_2(g_2)) \cdots (p_2(g_m))
\]
\[
\vdots
\]
\[
(p_n(g_1))(p_n(g_2)) \cdots (p_n(g_m))
\]

are all distinct, i.e. constitute all of \( G \).

If such a group \( G \), compounded in this way from \( m \) of its elements, exists, we call \( G \) a compound group.

Problem: Which (if any) finite groups are compound?
Steven H. Cullinane

Definition: Let $\mathcal{P}$ be a group of $n$ permutations on a finite $m$-element set $X$, let $G$ be a multiplicative group, and let
$f$ map $X$ to $G$.
Let $\mathcal{P} = \{p_i : 1 \leq i \leq n\}$, let $X = \{x_j : 1 \leq j \leq m\}$.
Define $\phi : \mathcal{P} \to G$ by $\phi(p_i) = \prod_{1 \leq j \leq m} f(p_i(x_j))$.
The structure $(\mathcal{P}, G, f, \phi)$ is a group compound.

Problem:
(1) Let $G$ be given. For which $\mathcal{P}$ and $f$ is $\phi(\mathcal{P})$ a coset of some subgroup of $G$? When is $\phi$ a surjection?
(2) Let $\mathcal{P}$ be given. For which $G$ and $f$ do inverse images under $\phi$ form a coset decomposition of $\mathcal{P}$? When is $\phi$ an injection?
Steven H. Callinane  
Table groups. Problem. November 27, 1983.

We regard the (unordered) tables of groups of order n as nxn arrays over the symbols 1, 2, ..., n in which the first row (read from left to right) is the same as the first column (read from top to bottom).
(The entry at top left represents the identity but need not be the symbol 1.) Thus we regard each of the arrays

\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & 1 & 4 \\
3 & 4 & 1 \\
4 & 3 & 2 \\
\end{array}
\hspace{1cm}
\begin{array}{ccc}
3 & 1 & 2 \\
1 & 3 & 4 \\
2 & 4 & 3 \\
4 & 2 & 1 \\
\end{array}
\]

as a table of the four-group.

Such a table is determined (since it is a Latin square) by the entries lying below the first row and to the right of the first column. Call this \((n-1)x(n-1)\) portion of a group table an n-box.

(Note that the number B of n-boxes is in general greater than \(nN\), where \(N\) is the number of nonisomorphic groups of order n. For instance, for \(n=4\) we have \(N=2\), but \(B=145\) rather than \(141\).)

For a given n, we may be able to see something of how the various order-n groups are interrelated by studying group actions on n-boxes.

Definition: Let \(G(n)\) denote the direct product of \((n-1)^2\) copies of \(S_n\) and regard the components of an element \(g\) of \(G(n)\) as arranged in an \((n-1)x(n-1)\) array. Such an element \(g\) acts componentwise in the obvious way on an n-box to yield an array that may or may not be an n-box.

Suppose there exists some subgroup \(T\) of \(G(n)\) such that \(T\) is transitive on some set \(B(T)\) of n-boxes that includes \(N\) n-boxes representing the \(N\) distinct (i.e., pairwise nonisomorphic) groups of order n. (We do not require that \(B(T)\) be closed under the action of \(T\), nor even that each \(T\)-image of a member of \(B(T)\) be an n-box.) We call such a \(T\) a table group for n.

Clearly for each n there is at least one table group \(T\), namely \(G(n)\). That smaller \(T\)'s may exist is shown by the following.

Example: A table group for 4 is generated by the following elements of \(G(4)\).

\[
\begin{array}{cccc}
1 & (12) & (12) & (13) \\
(12) & (12) & (13) & (11) \\
(12) & (12) & (13) & (11) \\
\end{array}
\hspace{1cm}
\begin{array}{cccc}
(13) & (13) & (13) & (11) \\
(13) & (13) & (13) & (11) \\
(13) & (13) & (13) & (11) \\
\end{array}
\hspace{1cm}
\begin{array}{cccc}
(11) & (11) & (11) & (11) \\
(11) & (11) & (11) & (11) \\
(11) & (11) & (11) & (11) \\
\end{array}
\]

Problem: What is the order of a smallest table group for n? Is there some way to construct such groups that does not require knowledge of \(N\)? (The case for \(n\) a prime power seems of particular interest.)

*The example is of course not a smallest table group for 4, but is shown for its structural interest.*
Let $X$ be a $2n$-dimensional linear space over a field $K$. Let a map $c$ take each subspace $S$ of $X$ to a function $f_S : X \to K$ that is nonzero on $S$ and zero elsewhere. (Here $f_S$ is a sort of characteristic function representing the subspace $S$.) Denote by $F = F(2n, K, c)$ the linear space over $K$ spanned by the functions $f_S$.

We call $F$ a geometric function space.

Theorem: There is at least one $F(2n, K, c)$ for which there exists a linear operator $T$, acting on $F$, such that $T^r$ takes the $l$-dimensional subspaces of $X$ (i.e., functions $f_S$ representing such subspaces) to distinct $r+l$-dimensional subspaces of $X$, for $1 \leq r \leq 2n-2$.

**Proof:** The matrix $T$ at right represents such an operator when $X$ is the linear 4-space over GF(2), the two-element Galois field.

Problem: For what other spaces $F(2n, K, c)$ does such a $T$ exist?
Module color-interchange and rotations, there are exactly 2 ways (see fig. 2) to color the 6 faces of a cube so that 
(a) each face is split diagonally into a black half and a white 
half, and 
(b) there are exactly 4 distinct images of the colored cube 
under the group 0 of 24 rotational symmetries of the cube.

The rotational symmetries of each such coloring form an order-6 
subgroup of 0 leaving invariant an inscribed hexagon as in fig. 1. 
This subgroup of 0 consists of the identity, rotations of 120 and 240 
degrees about a diagonal of the cube, and 180-degree rotations about 
each of 3 axes joining midpoints of opposite edges of the cube.

![Fig. 1: "Diamond" and "whirl" cubes]

Identical copies of these cubes, variously oriented, can be 
assembled into larger cubical patterns with remarkable symmetry 
properties.

![Fig. 3: A: Eight diamond cubes  B: Eight whirl cubes]

Patterns A and B in fig. 3 yield a number of other symmetric 
patterns when their subcubes are permuted (without rotation) as 
follows.

![Fig. 4:]

Let $S_4$ act on the 4 1x1x2 "bricks" in each of the 3 partitions 
above; the group $\Delta$ so generated can be shown to be triply transitive, 
of order 1344, and isomorphic to the affine group on the linear 
3-space over the two-element finite field.

THEOREM: Patterns A and B each have 168 images under $\Delta$. Each of 
these images has some nontrivial symmetry (ordinary symmetry for 
A-images, ordinary or color-interchange symmetry for B-images) under 
at least one of a group of 8 rigid motions of the cube.

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Theorem:
The affine group of order is generated by $S_4$ acting on partitions

<table>
<thead>
<tr>
<th>$AGL(n,2)$</th>
<th>of order</th>
<th>$S_4$ acting on partitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AGL(2,2)$</td>
<td>24</td>
<td>A</td>
</tr>
<tr>
<td>$AGL(3,2)$</td>
<td>1,344</td>
<td>A, B, C</td>
</tr>
<tr>
<td>$AGL(4,2)$</td>
<td>322,560</td>
<td>A, 2, 3</td>
</tr>
<tr>
<td>$AGL(5,2)$</td>
<td>319,979,520</td>
<td>A, B, C, 2, 3</td>
</tr>
<tr>
<td>$AGL(6,2)$</td>
<td>1,290,157,524,610</td>
<td>A, B, C, 1, 2, 3</td>
</tr>
</tbody>
</table>
"The typical example of a finite group is $\text{GL}(n,q)$, the general linear group of $n$ dimensions over the field with $q$ elements." -- J. L. Alperin

The 18 actions of $\text{GL}(2,3)$ on a $3 \times 3$ coordinate-array $A$ are illustrated above. The matrices shown right-multiply the elements of $A$, where

$$A = \begin{pmatrix} (1,1) & (1,6) & (4,2) \\ (4,1) & (2,6) & (1,2) \\ (2,1) & (4,6) & (6,2) \end{pmatrix}.$$ 

Actions of $\text{GL}(2,p)$ on a $p \times p$ coordinate-array have the same sorts of symmetries, where $p$ is any odd prime.
Steven H. Gullinane


The 48 diagrams below illustrate some symmetries of $GL(2,3)$ actions on the 6 nonzero vectors of the linear 2-space over the 3-element field. The vectors are viewed as labeling vertices of a cube (picted here with a slight distortion, to avoid overlapping lines).

The diagrams may have some heuristic value for the study of groups generated by mixing $GL(2,3)$ actions with those of other groups.
Two ways of partitioning a 72-set:

Definition: Let $G$ be the group of degree 72 generated by mixing

(1) actions of the affine group $AGL(2,3)$ on the set of nine $2 \times 2 \times 2$ cubes in partition $A$,

(2) like actions of $AGL(2,3)$ on each of the eight $3 \times 3$ sections in $B$,

(3) actions of $AGL(3,2)$ on the set of eight $3 \times 3$ sections in $B$, and

(4) like actions of $AGL(3,2)$ on each of the nine $2 \times 2 \times 2$ cubes in $A$.

Problem: What is the order of $G$?

Query: Clearly many similar problems could be posed.

What results or methods are known?

(Note: many equivalent coordinate systems for the affine actions above are available via natural mappings of the respective linear spaces onto $3 \times 3$ or $2 \times 2 \times 2$ arrays.)
Steven H. Gillilan


\[
\begin{array}{cccc}
0 & 1 \\
\times & \times+1 \\
\times^2 & \times^2+1 \\
\times^3 & \times^3+1 \\
\end{array}
\]

GF(3) A Singer 7-cycle S_3 on GF(3)
(mod \(x^2-x-1\)) 3_1 on GF(3)

The linear 4-space A linear map \(S_2\) on \(L\)

\(L\) over GF(2) \(= 2\) copies of \(S_1\)

\(S_2\) and \(S_2\) acting on row 1 below yield the Miracle Octad Generator [3]:

Apart from its use in studying the 759 octads of a Steiner system
\(S(5,8,24)\) -- and hence the Mathieu group \(M_{24}\) -- the Curtis MDO nicely
illustrates a natural correspondence \(C\) (Conwell [2], p. 72) between

(a) the 35 partitions of an 8-set \(H\)

(such as GF(3) above, or Conwell's \(H^\prime\) "heptads") into two 4-sets, and

(b) the 35 partitions of \(L\) into four parallel affine planes.

Two of the \(H\)-partitions have a common refinement into 2-sets iff
the same is true of the corresponding \(L\)-partitions. (Cameron [1], p. 60)

Note that \(C\) is particularly natural in row 1, and that partitions 2-5
in each row have similar structures.

2. Conwell, G.H., The 3-space \(PG(3,2)\) and its group,
   Ann. of Math. 11 (1910) 50-78.
3. Curtis, R.T., A new combinatorial approach to \(M_{24},
Steven H. Cullinane

The quintuply transitive Mathieu group $M_{12}$ might be expected to thoroughly scramble any neat pattern it acts on. However, recent work by R. T. Curtis and J. H. Conway [1] has the following remarkable consequence.

Theorem: The set of 7 infinite plane patterns below is invariant (modulo rigid motions of the plane, and color-interchange) under Curtis-Conway $M_{12}$ actions on the 4x3 motifs shown as quadrants.

Note that each pattern has nontrivial symmetry, modulo color-interchange. (The motifs are 7 of the 132 hexads in an $S(5,6,12)$ ingeniously constructed in [1].)

REFERENCE

Finite groups of the same order are sometimes related by a nontrivial identity.

Example:

\[\begin{array}{|c|c|c|c|c|}
\hline
+ & a & b & c & e \\
\hline
a & a & b & c & e \\
b & b & c & e & a \\
e & e & e & e & e \\
\hline
\end{array}\]

We have, \(\forall w, x, y, z \in \{e, a, b, c\}\),

(D) \(x^w(y^z) = (x^w)^y + (x^w)^z + x\), and hence

(I) \(((w+x)^y + (w+y)^z) = ((w+y)^x + (w+y)^z) = ((w+z)^x + (w+z)^y)) = ((w+z)^x + (w+z)^y)).\)

The dual identity I' obtained by interchanging + and * in (I) also holds.

Such a structure -- two groups joined together by a nontrivial identity -- might be called a "bridge." Are there infinitely many sorts of bridges? I am grateful to S. Comer for the following reformulation of this rather vague question.

Definitions: Let \(B = \{(G, *, ^, e): (G, *)\) and \((G, ^, e)\) are groups\}.

For a subvariety \(V \subseteq B\) let \(\Delta_v\) denote the set of identities holding in \((G, *)\) for all \((G, *, ^, e) \in V\). Similarly, define \(\Delta_v\). For any set of identities \(\Delta\) in the language for \(B\) let \(V(\Delta)\) denote the variety of all members of \(B\) that satisfy \(\Delta\). Call a variety \(V\) reducible if \(V = V(\Delta_v) \cap V(\Delta_v)\).

Problem: Are there infinitely many irreducible subvarieties of \(B\)?
S. H. Cullinane  

(A) Observation -- Nonisomorphic order-n groups, each transitive in permuting the same n points, may generate a group smaller than $A_n$.
Example -- The four group and $C_4$, acting on the vertices of a square, generate $D_4$.

(B) Observation -- Nonisomorphic order-n groups are sometimes related by a nontrivial identity.
Example --
\[
\begin{array}{cccc}
\text{r} & a & b & c \\
\text{c} & e & a & b \\
\text{a} & a & e & c \\
\text{b} & b & c & e \\
\text{c} & c & b & a \\
\end{array}
\quad \begin{array}{cccc}
\text{r} & a & b & c \\
\text{c} & e & a & b \\
\text{a} & a & b & c \\
\text{b} & b & c & e \\
\text{c} & c & e & a \\
\end{array}
\]

with \( x(yz) = (xy) + (xz) + x \ \forall x, y, z \in \{a, b, c\} \).

Problems:

(A) For which \((n,k)\) are there \(k\) nonisomorphic order-\(n\) groups \(G_i\) (each with the same elements and the same identity element) and regular permutation representations \(f_i\) such that \(\langle f_1(G_1), \ldots, f_k(G_k) \rangle < A_n \) ?

(B) For which \((n,k)\) are there \(k\) nonisomorphic order-\(n\) groups \(G_i\) (each with the same elements and the same identity element) all interrelated by a nontrivial algebraic identity?

(C) For which \((n,k)\) are there solutions to both (A) and (B)?
Definitions:

Given \( 0 < a \in \mathbb{R} \), and a finite (or countably infinite) sequence \( \xi = (a_1, a_2, \ldots) \) of positive real numbers such that \( \sum a_i = a \) (or such that the partial sums of \( \xi \) converge to \( a \)), call \( \xi \) a partition of \( a \). Let \( L(\xi) \) be the following surface:

\[
L(\xi) = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0 \text{ and } (xa_i)^2 = \sum (xa_i)^2 \}.
\]

Thus \( L \) is a mapping that lets us represent partitions by surfaces. (If the partial sums of \( \xi \) diverge but the corresponding surfaces converge, one might define \( L(\xi) \) to be the limit surface.)

Theorem (Nicomachus-Bachet):

The surfaces \( L((1, 2, \ldots, n)) \) all intersect at \((1, 2, 3)\).

Problems:

1. Do any other "natural" families of partitions yield intersection theorems of a nontrivial nature?

2. How do families of infinite-series partitions behave under \( L \)? (For example, \( \xi_5 = (1^{-s}, 2^{-s}, \ldots, n^{-s}, \ldots) \), for \( s > 1 \).)

3. Is the generalization of \( L \) by taking \((x, y, z) \in \mathbb{C}^3 \) impossibly difficult?
S. H. Gullinane


This note suggests a way to visualize the finite geometries recently described by A. Betelspacher in an excellent expository article [1].

Notation -- hexadecimal characters for the 15 points of PG(3,2):

\[ \begin{align*}
1 &= \text{OOG1} & 4 &= \text{0100} & 7 &= \text{0111} & A &= \text{1010} & D &= \text{1101} \\
2 &= \text{O010} & 5 &= \text{0101} & 8 &= \text{1000} & B &= \text{1011} & E &= \text{1110} \\
3 &= \text{OO11} & 6 &= \text{0110} & 9 &= \text{1001} & C &= \text{1010} & F &= \text{1111} .
\end{align*} \]

Facts about PG(3,2), the projective 3-space over GF(2):

(A) Each of the 15 points may be expressed as a sum of a unique pair of points from the set \( S = \{1,2,3,4,5,6\} \).

(B) Each of the 15 lines of PG(3,2) are distinguished by the fact that their points arise from partitions of \( S \) of the form \( 2+2+2 \); e.g., \( S = \{1,2\} \cup \{3,4\} \cup \{5,6\} \) yields the line \( \{3,4,7\} = \{1+2, 3+6, 4+5\} \). (The remaining 20 lines arise from partitions of \( S \) of the form \( 3+3 \), by summing pairs in the 3-sets.)

(C) Six spreads, each consisting of 5 mutually skew (i.e., disjoint) lines, can be formed from the 15 distinguished lines in (B).

These facts can be expressed graphically as follows.

(A)

(B) 15 distinguished lines. (Called an "encape" because of part-whole relationship.)

(C) The 6 spreads in (B). (Note correspondence with \( S \) in (A).)

Betelspacher describes a construction of PG(2,4) with

\( 21 \) points = the 6 points of \( S \) and the 15 distinguished lines (B), and

\( 21 \) lines = the 6 spreads (C) and the 15 point-pairs (A).

REFERENCE


This is the relativity problem: to fix objectively a class of equivalent coordinatizations and to ascertain the group of transformations $S$ mediating between them.

-- H. Weyl, The Classical Groups, Princeton Univ. Pr., 1946, p. 16

In finite geometry "points" are often defined as ordered \( n \)-tuples of elements of a finite (i.e., Galois) field $GF(q)$. What geometric structures ("frames of reference," in Weyl's terms) are coordinatized by such \( n \)-tuples? Weyl's use of "objectively" seems to mean that such structures should have certain objective -- i.e., purely geometric -- properties invariant under each $S$.

This note suggests such a frame of reference for the affine 4-space over $GF(2)$, and a class of $322,560$ equivalent coordinatizations of the frame.

The frame: A \( 4 \times 4 \) array.

The invariant structure:

The following set of 15 partitions of the frame into two $S$-sets.

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1
\end{array}
\]

A representative coordinatization:

\[
\begin{array}{cccc}
0000 & 0001 & 0010 & 0011 \\
0100 & 0101 & 0110 & 0111 \\
1000 & 1001 & 1010 & 1011 \\
1100 & 1101 & 1110 & 1111 \\
\end{array}
\]

The group: The group $AGL(4,2)$ of $322,560$ regular affine transformations of the ordered $4$-tuples over $GF(2)$.  

32
S. H. Cullinane


If a group $G$ acts on a set $X$, there is a natural closure operation
on subsets of $X$: define topological closure as closure under
$G$-actions. Then the closed sets (in both senses) are the empty set,
the $G$-orbits, and arbitrary unions of $G$-orbits. ($A \subseteq X$ is open iff
$A$ is closed.) The result is a group topology $T(G, X)$.

Unfortunately, $T(G, X)$ is trivial if the group action is transitive.
But $G$ acts on the power set $P(X)$ as well as on $X$, and we have

1. $X$ is nonempty $\Rightarrow T(G, P(X))$ is not trivial, and
   the $G$-action is nontrivial $\Rightarrow T(G, P(X))$ is not discrete and not $T_1$
   (i.e., not all singletons are closed).

(That a topology is not $T_1$ is unfortunate if the underlying set is
infinite, but very fortunate if the underlying set is finite.)

Let $P_0 = X, P_n = P(P_{n-1}(X))$, and let $T_n = T(G, P_n(X))$.

Problems:

1. Is there a purely set-theoretic characterization of the finite $T_n$
   (i.e., among all other topologies on $P_n$ based on partitions that
   refine the cardinality partition)?

2. Consider the topologies $T_n$ for a faithful action $F$ of $G$ on $X$.
   (a) Is $F$ always determined by $T_0, T_1, \ldots, T_n$ for some $n = n(F)$?
   (b) If $H \leq G$, how are the $T_i$ for $H$ related to the $T_i$ for $G$?
   (c) If $X$ is countably infinite, can we regard the minimal closed sets
       of $T_1$ as "natural" $G$-orbits on some continuum?
S. H. Cullinane
Picturing the smallest projective 3-space. April 26, 1986.

The 2 figures at left show a symplectic polarity A: each point lies in its corresponding hyperplane. The 15 lines fixed under A are shown in fig. A below.

The 15 points

A

The 15 hyperplanes

B

C

The 35 lines

The 6 spreads in A

Sums of the 1-subsets of A pictured in A

Sums of the 2-subsets of A pictured in B or C

The F. T. Curtis correspondence between the 35 lines and the 2-subsets and 3-subsets of a 6-set. This underlies $M_{24}$.
S. H. Cullinane

Figures A, B, C show the 35 lines of $PG(3,2)$; fig. A is a linear complex.

Figures $A', B', C'$ show the R.T. Curtis correspondence between the 35 lines and the 35 partitions of an 8-set into two 4-sets. This underlies $M_{24}$. 
J. H. Cullinane

The 2-subsets of a 6-set are the points of a PG(3,2). May 26, 1986.

This note was suggested by
(1) A. Beutelspacher's model [1] of the 15 points of PG(3,2) as the
15 partitions of a 6-set into three 2-sets, and by
the 35 lines of PG(3,2) and the 35 partitions of an 8-set into
two 4-sets.

If \( X \) is a finite set, we may regard the power set \( P(X) \) as an
elementary abelian 2-group in which addition is the set-theoretic
symmetric-difference operation. Let \( K(X) \) be the subgroup
of \( P(X) \) consisting of \( \emptyset \) and \( X \), and let \( Q(X) = P(X)/K(X) \).

When \( X \) is a 6-set, the 2-subsets form a subgroup \( A \) of \( Q(X) \) whose
nonzero elements we may take as the points of a PG(3,2), with
collinearity defined in the obvious way.

A subgroup of \( Q(X) \) illustrating
(1) the 15 2-subsets of a 6-set
(2) the 15 points of PG(3,2)

Subsets of \( A \) illustrating
(1) the Curtis correspondence between
\( A \)-sets and the 15 partitions of a
6-set into three 2-sets
(2) a linear complex in PG(3,2)

REFERENCES

1. Beutelspacher, A., 21 - 6 = 15: A connection between two

2. Conwell, G. H., The 3-space PG(3,2) and its group,
Ann. of Math. 11 (1910) 60-76 (esp. p. 72).

3. Curtis, R. T., A new combinatorial approach to \( M_{24} \),
S. H. Gullinane

Shown below are the 21-point projective plane PG(2,4) and its dual. The points (or lines) are the 21 partitions of a 6-set into disjoint sets A, B, where |A| = 2 or 1.

Lines (or points):  
\[\text{Points (or lines):}\]

Points on the above lines (or lines on the above points):

The 6-set permutation interchanging points and lines is from the Miracle Octad Generator of R. T. Curtis [1, p. 28].

REFERENCE
S. H. Cullinane

Figure A below shows the 2-subsets of a 6-set $S$; figure B shows the locations in $A$ of the triples of 2-subsets that partition $S$.

The map $C$:

Together, $A$ and $B$ specify a correspondence $C$ between the 15 subsets and the 15 partitions. This correspondence leads in a natural way to

1. a model of the projective plane $PG(2,4)$ in which the 21 points (and also the 21 lines) are the 21 partitions of $S$ into subsets $X$, $\bar{X}$, where $|X| = 2$ or 1;
2. the Conwell mapping of the $35 \ (4+4)$-partitions of an 6-set onto the 35 lines of $PG(3,2)$, which preserves certain intersection properties;
3. the R. T. Curtis "MOG" model of the Steiner system $S(5,8,24)$ and of $M_{24}$ as the model's automorphism group.

Let $f:s \rightarrow s'$ exchange rows 2 and 3 in each $3 \times 2$ picture $s$ in $A$, and let $C'$ map a 2-subset $s$ to $C(s')$. If we regard the 2-sets and partitions as transpositions and products of transpositions, $C'$ induces an outer automorphism $p$ of $S_6$. (In the above $PG(2,4)$, $S_6$ and $p(S_6)$ act in concert as a group of collineations.)
Shown above are two ways to picture some outer automorphisms of $S_6$ that have been discussed in the literature (θ in (1), φ in (2)). In the top row, figure X shows the 15 2-sets in a 6-set $S$, and θ, φ show the locations in $X$ of triples of 2-sets that partition $S$. The second row shows the corresponding permutations.

Each row's θ, φ contain 6 special 5-subsets:

\[
\begin{array}{cccc}
\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \\
\boxed{5} & \boxed{6} & \boxed{7} \\
\end{array}
\]

In the top row these 5-subsets are spreads of lines in a PG(3, 2); in the second row they are parallelisms of $S$. Such 5-subsets (each of which can be selected in 6 ways, then arranged in 5! ways) determine the 61 outer automorphisms of $S_6$.

REFERENCES


S. H. Cullinane

This note was suggested by J. H. Conway's construction (1) of an order-2 outer automorphism of $S_6$.

Figures A and B above each show 16 permutations of a 16-set that generate groups $G(A)$ and $G(B)$, respectively. Figure X shows 16 subsets of a 16-set. The groups $G(A)$ and $G(B)$ can act on figure X in two ways: by an inner action on each of the 16 $4 \times 4$ parts individually, or by an outer action permuting the 16 parts.

Theorem: Let $a$ denote any permutation in A, and let $b$ denote the permutation in the corresponding location in B. Then the inner (outer) action of $a$ on X induces (is induced by) the outer (inner) action of $b$ on X. The group $G(A)$, and hence $G(B)$, is isomorphic to $S_6$, and the map taking each $a$ to its corresponding $b$ extends to an involutive outer automorphism of $S_6$.

REFERENCE